ACTEX Academic Series

Probability & Statistics with Applications:

A Problem Solving Text

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Mark M. Maxwell, Ph.D., ASA

2

Second Edition

PROBABILITY AND STATISTICS WITH APPLICATIONS:

A PROBLEM SOLVING TEXT

SECOND EDITION

For Marilyn, - L.A.

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PREFACE

Preface to the Second Edition

This edition contains a number of new topics, primarily in the mathematical statistics portion of the text. We continue to cover all of the material necessary for the SOA Exam P (Probability) in Chapters 1-8. We have expanded the second part of the book, now Chapters 9-12, to provide full coverage of the syllabus for the Casualty Actuarial Society Exam ST, and the statistics portion of the new CAS Exam S.

New sections include explanations and worked examples on the following topics:

- Mixture distributions,
- Non-homogeneous Poisson processes,
- Sufficient statistical estimators and the linear exponential family,
- Bayesian analysis and conjugate prior distributions,
- Nonparametric statistical methods,
- Graphical methods.

As in the first edition, our aim is to provide the best qualities of both a standard textbook and an actuarial exam study manual. We attempt to provide full explanations and derivations or proofs wherever feasible for an introductory text. We also provide a multitude of examples and problems, including many from earlier actuarial exams. Space limitations prevent us from providing complete solutions to all of the exercises in the body of the text. However, a complete solutions manual is available as a separate volume.

We have benefited from much assistance from reviewers and editors for this edition. In particular we express our appreciation to Ali Ishaq, Tom Lonergan, Rajesh Sahasrabuddhe and Emiliano Valdez, who read all or portions of the manuscript and provided valuable suggestions.

We are indebted to David Hudak for his assistance with the new material on non-homogeneous Poisson processes.

We also gratefully acknowledge the detailed editorial review provided by Geoff Tims. Finally, we extend heartfelt thanks to Stephen Camilli and Garrett Doherty of ACTEX, the former for his editorial guidance and encouragement, the latter for his formidable technical skills in producing the final manuscript pages.

L.A.

Preface to the First Edition

A cursory search of Google Books reveals thousands of titles with the words *Probability and Statistics*, or *Mathematical Statistics*, thus prompting the inevitable question, why yet another? In view of this superabundance of choices we feel impelled to offer a few words of explanation in our defense for adding to this already crowded marketplace.

The idea for this textbook evolved out of a two-semester course in probability and statistics we have been offering for many years to sophomore and qualified freshman students, predominantly actuarial science majors. The goals of our course are twofold: to lay the foundations of calculus-based probability theory for our students, and to prepare them to pass the actuarial exam covering this material as early as possible in their collegiate careers.

The primary market segment we are targeting consists of students like ours – freshmen and sophomores – who are studying calculus-based probability while simultaneously learning the selfsame calculus it is based upon. We desire that our actuarial students be prepared to pass Exam P/1 (jointly offered by the Society of Actuaries of Actuaries (SOA) and the Casualty Actuarial Society (CAS)) no later than the end of their sophomore year. Consequently, the probability and statistics component of their education tends to overlap with the typical 3-semester calculus sequence they all take.

Our experience has been that the myriad existing textbooks in probability and statistics fall into two types. They are either designed to support a “calculus-free” environment suitable for general business school statistics courses, or they are intended for more advanced, calculus-based mathematical statistics courses for juniors and seniors with the requisite technical background. The first category does not provide the depth of understanding required for the actuarial exam, and the second type of book tends to be too formal and advanced for our students.

For these reasons we decided to produce an introductory text in calculus-based probability and statistics whose level is comparable to a modern-day calculus book, and that could reasonably be used by freshman or sophomore students studying the material concurrently with their calculus classes. We consciously strive to pace the material in a way that makes it accessible to a student whose background consists of just one semester of college-level calculus. This might be, for example, either entering freshmen with AP or high-school/college credit concurrently taking Calculus II, or sophomores with one or two semesters of calculus already under their belts.

Chapters 1-4 present the rudiments of probability theory for discrete distributions with little or no reference to calculus topics, save for the basic knowledge of infinite series required for understanding the geometric and the Poisson distributions. Chapter 5, entitled Calculus and Probability, introduces continuous random variables and is the first chapter heavily dependent on derivatives and integrals. The material on continuous, jointly distributed random variables comes in Chapter 7, by which time our students will have been introduced to double integrals in their Calculus III class. The second part of the book, comprising Chapters 9-11, covers all of the syllabus topics for the statistics portion of CAS Exam 3L – Life Contingencies and Statistics Segment. Taken as a whole, this book provides ample content to serve as the text for the standard two-semester introductory sequence in mathematical statistics and probability.

The text contains nearly 800 exercises, many with multiple parts. Numerical answers are given in the back of the book and a supplementary manual with complete solutions is available separately. Many of the exercises, and some examples, are based on previous actuarial exam questions released by the SOA and the CAS. All of the SOA/CAS Exam P/1 Sample Exam Questions (142 in number at press time in late 2009) have been incorporated into the text. In addition, we have used many statistics questions from CAS Exam 3 and 3L, as well as questions from the earlier Exam 110 (Probability and Statistics), in Chapters 8-11. We are grateful to the Society of Actuaries and the Casualty Actuarial Society for their permission to use these materials.

While designed primarily with the actuarial audience in mind, we hope this text will have appeal for a broader audience of mathematics and statistics students. We have sought to engage students with a light, informal style, emphasizing detailed explanations and providing a multitude of examples. At the same time, we have sought at each stage to present a sufficient glimpse into the theoretical underpinnings to make the text suitable for more advanced students. The overarching leitmotif, however, is problem solving, and we emphasize the requisite skills throughout. We hope to have a struck a balance that will allow students at all levels to benefit from a close reading of the text.

We have benefited from the many helpful comments and valuable insights provided by our reviewers: Carolyn Cuff, Westminster College; Thomas Herzog, Department of Housing and Urban Development; Thomas Lonergan, CIGNA; Jeffrey Mitchell, Robert Morris University; Emiliano Valdez, University of Connecticut; Charles Vinsonhaler, University of Connecticut. We would also like to acknowledge the invaluable editorial assistance provided by Gail Hall of ACTEX, whose firm but gentle hand managed to guide this project to completion. We are also indebted to Marilyn Baleshiski, whose patience and skill rendered a lumpy manuscript into a finished text. Our thanks to you all. Needless to say, for all remaining errors and indiscretions, the authors have only each other to blame.

Finally, we note that personages (both named and unnamed) who appear in various exercises and examples are completely fictitious and any resemblance to real people (living or departed) is purely coincidental.

Len Asimow, Moon Township, PA
Mark Maxwell, Austin, TX

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CHAPTER 1

COMBINATORIAL PROBABILITY

1.1 THE PROBABILITY MODEL

The fundamental object of interest in this text is the ***probability model***. Probability models are based on experiments, such as flipping a coin or tossing a pair of dice, for which there are multiple possible outcomes. It is impossible to say in advance which outcome will occur. In this sense a probability model constitutes an apt metaphor for life, where we rarely know the ultimate results, or future outcomes, of our actions. This is in contrast to *deterministic* models, such as a well-designed chemistry experiment, where one knows in advance the outcome with a high degree of certainty (a bad odor will arise, a small explosion will occur, a penny will completely dissolve in 60 seconds, etc.).

A probability model has three essential elements. These three elements can be defined as follows:

The Elements of a Probability Model

i. The description of the underlying probability experiment,
ii. A list, or a procedure for listing, all possible ***outcomes*** of the experiment,
iii. A rule for assigning numbers between 0 (impossible) and 1 (certain) to subsets of outcomes.

The set of all outcomes on the list in (ii) is called the ***sample space***. Subsets of the sample space are called ***events***. The numbers in (iii) are called ***probabilities***, and they measure the likelihood of an outcome along a scale from 0% to 100%. These numbers must satisfy one other property, denoted below as the ***additive property***, which we will consider in more detail as we continue.

The Additive Property

iii'. The “sum” of the probabilities of all the outcomes must equal one.

“Sum” is written in quotation marks because the summing process may be more complicated than simple addition, possibly involving an infinite series or a definite integral. As a consequence of the additive property, it seems sensible (and it is correct) to define the *probability of an* ***event*** as the "sum" of the probabilities of all the outcomes contained in the event. The additive property assures that the event consisting of the entire sample space has probability one. That is, it is certain that one of the outcomes listed in (ii) will occur.

Example 1.1-1 Elements of a Probability Model

The description of an experiment could be as simple as "*toss a coin once*." A complete list of outcomes would be the set consisting of *heads* or *tails*. A simple rule for assigning probabilities is $\frac{1}{2}$ for heads and $\frac{1}{2}$ for tails. Notation that summarizes these statements is

$$\text{sample space} = \{heads,\ tails\},\quad \Pr(heads) = \frac{1}{2},\ \text{and}\ \Pr(tails) = \frac{1}{2}.$$

In this case the experiment, the outcomes, and the probability assignments constitute an accurate model of the real-world activity of tossing a balanced (fair) coin. In life there is no such thing as a perfectly balanced coin, and there is always the remote possibility of a third outcome (e.g., *coin stands on edge*). But since we are constructing an idealized mathematical model of an experiment, we keep things as simple as possible while preserving the essential structure of the real world phenomena we are modeling.

Example 1.1-2 Sample Space

Suppose that a couple conducts the experiment of having three children. Their concern is the *gender* of the children.

a. List the sample space of all possible outcomes.
b. List the event that a boy is the first born child.

Solution

The sample space can be represented by the set $\{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$, where B denotes the birth of a boy and G denotes the birth of a girl. Therefore the *outcome* BGG denotes a boy being the first born, followed by the births of two girls. The *event* of a boy being the first born consists of the subset $\{BBB, BBG, BGB, BGG\}$.

Here, the set of outcomes was identified as a list of *words*. By *words* in this text we mean any systematic grouping of letters or numbers used to describe outcomes. In this example, the outcomes consist of all 3 letter *words* (not just dictionary words) consisting of the letters B and G exclusively. Note that the words are systematically listed in alphabetical order, which is not only convenient, but helps to assure that we don't inadvertently omit an outcome. ❒

If all experiments were as simple as tossing a single coin, or having three children (which is conceptually simple, although more complicated in actual practice[1]), then you wouldn't need this text. Needless to say, probability models can be quite complex, with a very large (or even infinite) number of outcomes.

---

[1] This comment was written by Len, father of two.

There are a number of different ways of classifying probability models, but the most basic is the distinction between *discrete* probability models and *continuous* probability models. In *discrete* probability models, the outcomes can be enumerated in a list, as *outcome*(1), *outcome*(2), etc. This list can be *finite* or *countably infinite*. An example of the latter would be the experiment of tossing a coin until the first head appeared. One way of listing this sample space would be,

$$\{H, TH, TTH, TTTH, \ldots\}$$

Alternatively and equivalently, we could describe the sample space as $\{1, 2, 3, 4, \ldots\}$, where the natural number stands for the coin toss on which the first head occurs. If this were a fair coin, then an appropriate assignment of probabilities would be $\Pr(H) = \Pr(1) = .5$, $\Pr(TH) = \Pr(2) = .25$, $\Pr(TTH) = \Pr(3) = .125$, and so forth. In Chapter 4 we will verify that the infinite sum of these probabilities equals one.

A *continuous* model has outcomes along a *continuum*. A description of such an experiment might be, "*toss a dart at the unit interval* $[0,1]$." The set of outcomes consists of all real numbers between 0 and 1. For those readers impertinently protesting that one can't toss a dart at a one-dimensional line in the real world, we can reformulate the experiment as "*spin a wheel-of-fortune*" whose edge is calibrated with the unit interval. Here, the fixed pointer plays the role of the dart, and we are now "*throwing the unit interval at the dart*," so to speak. In either case, the mathematical model is the same. The real difficulty here is in assigning the probability numbers. Since the model is continuous, it should not be surprising that we require the tools of calculus to do this properly. Essentially, probabilities are assigned to intervals, not single points, and the summing of the probabilities referred to in *additive property* (iii') requires *integration*. In this example, the probability of the pointer landing in the interval $[a,b]$ is its length, $b - a$. We will leave the details until Chapter 5.

Finally, it must be emphasized that the *way* in which the list in (ii) is constructed can be extremely important.

Example 1.1-3 Sample Space and Assigning Probabilities

Consider the experiment of *rolling a pair of fair dice.*

(a) List all outcomes in the sample space.

(b) Assign probabilities to the outcomes.

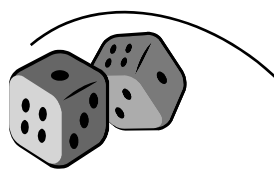
(c) Find the probability that the sum of the dice is 8.

Solution

Generally, when we roll a pair of dice the outcome of real interest is the *sum*. In particular, we would like to model the real-life probability of the outcome *sum equals* 8. Our list of all outcomes could therefore be the eleven integers from 2 to 12. That leaves the issue of assigning probabilities. Since this is a mathematical model we can assign probabilities any way we like so long as the probabilities sum to one. In particular, we could arbitrarily assign *equally likely outcomes* so that the probability of a sum equal to 8 would be the same as the probability of a sum equal to 12 or any other outcome, namely 1/11. However, this model will not conform to our experience gained over the many years of playing with dice.

We will therefore take a different approach to listing the outcomes. We first imagine in our mathematical model that we can distinguish between the two dice (assume that one is green and the other is grey). We then list all outcomes as two-letter *words* with the *letters* being the *integers* from 1 to 6), with the green die result followed by the grey die:

For example, the word (**1**4) represents the green die (left) showing one dot and the grey die (right) showing four dots.



11	**2**1	**3**1	**4**1	**5**1	**6**1
12	**2**2	**3**2	**4**2	**5**2	**6**2
13	**2**3	**3**3	**4**3	**5**3	**6**3
14	**2**4	**3**4	**4**4	**5**4	**6**4
15	**2**5	**3**5	**4**5	**5**5	**6**5
16	**2**6	**3**6	**4**6	**5**6	**6**6

In this second method of listing, it *is* reasonable to assume that the outcomes are all equally likely. This is because we assume that the individual die gives equally likely outcomes from 1 to 6 (such dice are called fair and balanced - like the ideal newscast), and therefore the pairs are also equally likely to occur. In this second method, there are a total of 5 outcomes out of 36 with a *sum equal to* 8. Thus, "*sum of 8*" is no longer an *outcome,* but an *event*. If the 36 words are all assigned the probability of 1/36, then the probability of the event *sum of* 8 is $\frac{5}{36}$, which conforms to actual experience. Even though there are more outcomes (36 versus 11) to deal with, the individual outcomes have the advantage of being naturally equally likely. This provides a mathematical model that conforms more readily to reality.

In general it is better to list outcomes in a way that preserves as much information as possible about the underlying experiment, and which takes advantage of an intuitively natural assignment of equally likely outcomes. ❐

Exercise 1-1 At a picnic, you select a main course from a hamburger or a chicken sandwich. You select a side from potato chips or coleslaw, and you select a beverage from soda, water, or milk. These 3 selections result in a **meal**.

(a) Write out the sample space of all **meals**.

(b) Write out the event (subset) consisting of all meals with soda as the beverage.

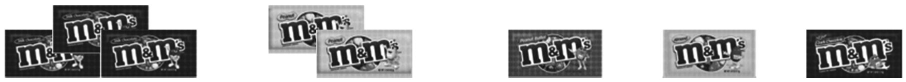
(c) Assuming all meals are equally likely, calculate the probability of getting soda with your meal.

Exercise 1-2 Roll a pair of fair six-sided dice.

(a) How many ways can *doubles* (both die have the same number of dots) be rolled?

(b) What is the probability that the *sum of the two dice is a prime number*?

Exercise 1-3 A convenience store has three packages of plain M&M'S®,[2] two packages of peanut M&M'S, one package of dark chocolate M&M'S, one package of peanut butter M&M'S, and one package of almond M&M'S. Select two packages of candy.



(a) Write out the sample space of all possible outcomes.

(b) Assume the outcomes are equally likely. Calculate the probability you select exactly one package of plain M&M'S, (and the other package is **not** plain – either peanut, dark chocolate, peanut butter, or almond).

(c) Calculate the probability of getting **at least** one package of plain M&M'S.

1.2 FINITE DISCRETE MODELS WITH EQUALLY LIKELY OUTCOMES

In this section we consider in more detail the wide class of probability models in which there is only a finite number of possible outcomes. Probabilities are assigned, as in the dice example in the previous section, so that all outcomes are equally likely. This situation of *equally likely outcomes* governs many of the typical probability problems that arise in modeling games (drawing cards, tossing dice, flipping coins, etc.).

In the case of equally likely outcomes, the assignment of probabilities is straightforward. Since the sum of the probabilities of outcomes must equal one, and the outcomes are equally likely, each outcome has probability $\frac{1}{n}$, where n is the total number of possible outcomes.

Probability with Equally Likely Outcomes

Let S be a sample space consisting of n equally likely outcomes. Let E be an event in S consisting of r outcomes. Then the probability of event E, denoted $\Pr(E)$, is

$$\Pr(E)=\frac{\text{number of outcomes in event } E}{\text{total number of possible outcomes}}=\frac{r}{n}=\frac{N(E)}{N(S)},$$

where $N(E)$ denotes the number of elements in the set E.

Determining the value of n or r is often the most complicated part. This leads us to a discussion of various counting techniques (called *combinatorics*) such as *permutations* and *combinations*.

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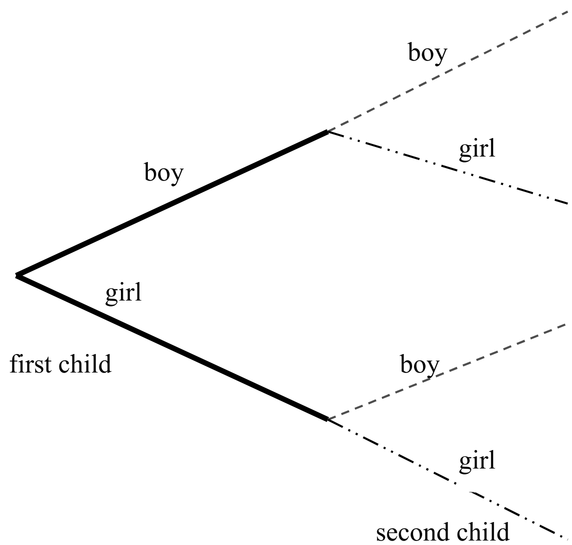
1.2.1 TREE DIAGRAMS

A tree diagram is a fundamental conceptual tool for constructing lists of outcomes. It is also a valuable approach for tracking probabilities in a multistage experiment.

Example 1.2-1 Tree Diagram

A family has two children. Construct a tree diagram showing all possible combinations of boys and girls.

Solution



There are four distinct possible outcomes, each corresponding to exactly one path through the diagram. Traveling across the top path, for example, denotes the birth of two boys. Using the diagram, we can write the sample space as $\{BB, BG, GB, GG\}$ where B denotes the birth of a boy and G denotes the birth of a girl.

Example 1.2-2 Tree Diagram of Your Wardrobe

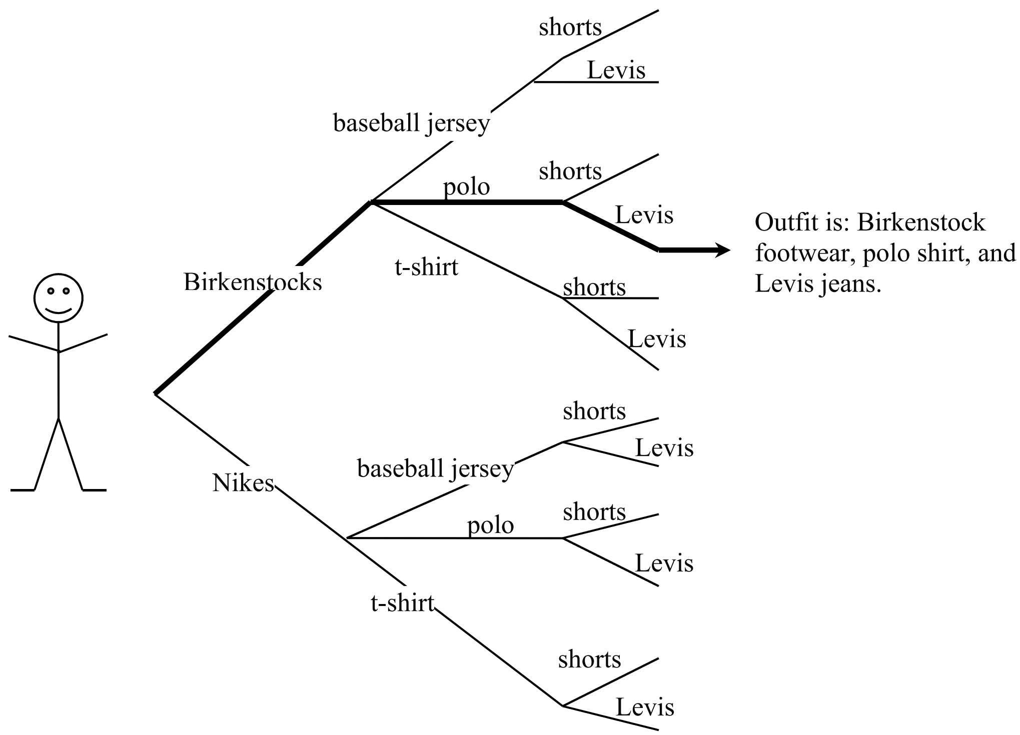
Suppose that your wardrobe consists of two types of shoes (Birkenstocks and Nikes), three different shirts (baseball jersey, polo, tee-shirt), and two types of pants (shorts and Levi button-fly 501 jeans).

(a) Construct a tree diagram showing all possible outfits that you could wear.

(b) Assuming all outfits are equally likely, calculate the probability of choosing an outfit that includes a polo shirt.

Solution

(a)



(b) The diagram consists of twelve distinct branches, each corresponding to a distinct outfit. Four of these branches include a polo shirt, so that the probability is 4/12 = 1/3. This is, of course, the same as the probability of choosing a polo shirt from among the 3 possible shirts available. ❐

In these examples, each *outcome* can be represented by a particular path through the tree. In the first example there are two possible results (*boy* or *girl*) for the gender of the first child, and each of these leads to two possible results for the gender of the second child. This leads to a total of $2 \times 2 = 4$ paths (outcomes) through the tree. In the second example we have 2 possible pairs of shoes, and for each of these there are 3 possible shirts. For each choice of shoes and shirt, there are 2 choices for pants. The total number of paths through the tree is $2 \times 3 \times 2 = 12$.

The general rule at work here is called the *multiplication principle*.

1.2.2 THE MULTIPLICATION PRINCIPLE

The Multiplication Principle

Suppose an experiment can be broken down into a first stage A consisting of $N(A)$ outcomes **and** that for each of these outcomes, there is a second stage B consisting of $N(B)$ outcomes. Then the total number of outcomes for the two stages combined is equal to $N(A)\times N(B)$.

Note: The word **and** is as indicator that one should multiply. The rule can be extended to three or more stages.

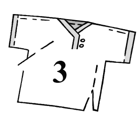
This principle is commonly called *the **fundamental theorem of counting***.

Example 1.2-3 The Multiplication Principle

For the wardrobe described in Example 1.2-2, use the fundamental theorem of counting to determine how many different outfits you could wear.

Solution

We used a tree diagram to list the sample space of all of the possible outfits. In this problem, our concern is only the number of possible outfits we have, so we use the fundamental theorem of counting.

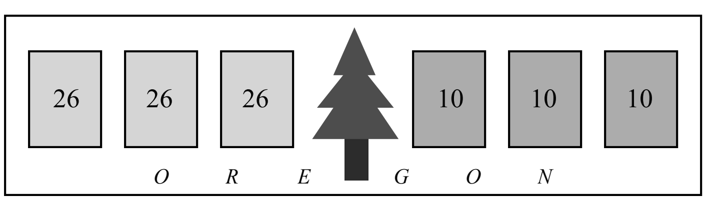
 **2** $\times$  **3** $\times$ **2**  $=$ 12. ❐

Example 1.2-4 The Multiplication Principle

Assume standard license plates in the state of Oregon are comprised of three letters followed by three numbers. How many different license plates could theoretically be fabricated?

Solution

The total number of standard license plates is $26\cdot 26\cdot 26\cdot 10\cdot 10\cdot 10 = 26^3\cdot 10^3 = 17{,}576{,}000.$ Using a tree diagram to list all possible outcomes from AAA000 through ZZZ999 would require more space than is available here.



❐

Exercise 1-4 Your current wardrobe consists of eleven different items: three pairs of shoes, two pairs of pants, and six different shirts. Your mom offered to buy you a new shirt **or** a new pair of shoes. Which item (shoes or shirt) should your mom buy for you to maximize your total number of outfits (shoes-shirt-pant)?

Exercise 1-5 Roll three fair six-sided dice.

(a) Assuming the dice are distinguishable and always tossed in a particular order (e.g., red-white-blue), how many different outcomes are possible?

(b) How many outcomes are there in which the first two dice come up 3 and 4 respectively?

(c) What is the probability that the first two dice come up 3 and 4 respectively?

(d) Would you like to use a tree diagram to list all of the possible outcomes?

Exercise 1-6 Assuming there are no restrictions, how many different 9-digit social security numbers are possible?

Exercise 1-7 Assume that a standard California license plate consists of a digit, 3 letters, and then 3 more digits (e.g., 1SAM123), except that I, O, and Q are excluded from the first and third alpha positions. How many possible standard license plates are there in California?

Exercise 1-8 A thief plans to rob four banks. His town of Podunk, Idaho has six banks. He could rob a bank twice, but never consecutively. In how many ways may he select the four banks?

Exercise 1-9 How many ten-digit phone numbers are possible in the United States? **Note**: Creating a mathematical model that accurately reflects the real world is extremely important and frequently difficult. Rules that govern the problem should be understood and listed. In this case, we assume that the first digit cannot be 0 or 1 and the fourth digit cannot be 0.

1.2.3 PERMUTATIONS

Permutations

Given a set of n distinguishable objects, an ordered selection of r different elements of the set is called a ***permutation of** n **objects chosen** r **at a time.***

Example 1.2-5 Permutations

In May you travel to Louisville and make a trifecta wager on the Kentucky Derby. A trifecta bet consists of selecting the first three finishers in order. That is, the one that finishes first (win), the one that finishes second (place), and the one that finishes third (show).. How many different trifecta wagers are possible if 14 horses go to post? Rephrased mathematically, how many permutations are there of 14 objects (distinguishable horses) chosen 3 at a time?

Solution

First we note that any of the 14 horses could win. But the winning horse cannot also finish second, so there remain 13 horses that could be selected for second. Similarly, there remain 12 horses that can be selected for third. Using the multiplication principle directly we see the number of distinct trifecta wagers is,

14	13	12	= 2184
Win	Place	Show	

❐

Factorials

Let n be a whole number. Then $n!$ (read as "n factorial") is defined by

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1.$$

By convention, we define $0! = 1$.

Alternatively, we can express the solution to Example 1.2-5 using factorial notation.

Using factorial notation we can easily generalize from the horserace example to a formula for the number of permutations.

Permutation Formula

The number of permutations on n objects chosen r at a time is denoted by ${}_nP_r$.

$${}_nP_r = \underbrace{n(n-1)(n-2)\cdots(n-r+1)}_{\text{first } r \text{ factors of } n!} = \frac{n!}{(n-r)!}.$$

Note: The $(n-r)!$ in the denominator cancels out the "tail" of $n!$, leaving just the first r factors.

Note: Other common ways to denote permutations: ${}_nP_r = P(n,r) = P_r^n$.

This formula results from applying the multiplication principle to the r stage process of picking the r objects, one at a time and in order.

Note

$_nP_n = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!} = n!$ is simply the number of distinct ways of rearranging (choosing a different ordering) all n objects.

The solution to the horserace problem can be rephrased as the number of permutations on 14 horses chosen 3 at a time,

$$_{14}P_3 = \frac{14!}{(14-3)!} = \frac{14!}{11!} = \underbrace{14}_{\text{win}} \cdot \underbrace{13}_{\text{place}} \cdot \underbrace{12}_{\text{show}} = 2{,}184.$$

We use permutations when the ***order*** of selection is important.

Exercise 1-10 Find the numerical values of the following permutations:

(a) $_7P_6$	(b) $_{13}P_{12}$	(c) $_7P_4$
(d) $_5P_2$	(e) $_4P_4$	(f) $_4P_6$
(g) $_{23}P_3$	(h) $_3P_0$	(i) $_5P_5$
(j) $_{10}P_1$	(k) $\dfrac{_7P_4}{4!}$	(l) $_0P_0$

Exercise 1-11 An engineering faculty member shuffles the quizzes of his 17 students and redistributes them randomly for peer grading.

(a) In how many different ways may the quizzes be handed out?
(b) In how many ways may Kendra, one of the 17 students, be assigned her own quiz for grading?
(c) What is the probability that Kendra will receive her own quiz?

Exercise 1-12 Peter Pickemfast spends a day at the races and decides to play the trifecta in an eight horse field. Knowing nothing about any of the horses, he chooses horses randomly and buys one ticket. Calculate the probability he has the winning trifecta ticket.

Exercise 1-13 Henry Hoodlum speaks to two of the jockeys in an 8 horse field, politely requesting they not finish in the top 3. Assuming they comply, how many trifecta tickets must Henry buy to assure he holds a winning ticket?

Exercise 1-14 A psychology professor has twelve students in class. She insults one student, throws an eraser at a second student, lowers the grade of a third student, and sends a fourth student to the dean's office. In how many different ways can she perform these motivational techniques?

1.2.4 THE BIRTHDAY PROBLEM AND ITS GENRE

There is a classic problem in probability that asks, "What are the chances that two people in this room share the same birthday (month and day only)?" The answer, of course, depends on how many people are in the room, and how many possibilities there are for birthdays. To keep things simple we assume that there are exactly 365 equally likely birthdays. Thus, if you happen to have been born on February 29, you don't count. We could let you celebrate on February 28, but then we would no longer have equally likely outcomes. It is easier to just assume you do not exist in our simplified model. You still exist in the real world and our model aspires to be a simple approximation to reality. We are also aware that birthdays are not equally likely throughout the year, but dealing with this peculiarity would further complicate our model.

We will illustrate the use of the multiplication rule and permutations to answer the question in a general format. Assume there are n distinguishable objects ("birthdays") and we wish to assign them to r "people." We will calculate the probability of the event, E, that there is *no* duplication of birthdays. The event that there *exists* a duplication of birthdays is called the *complement* of E and is denoted E'. Its probability must equal one minus the probability of the event E.

The sample space, S, consists of all possible ways of distributing n "birthdays" to r "people." This is an r stage process with n ways to choose at each stage. Thus, the size of the sample space equals n^r (the multiplication principle in action). Now, the event E consists of all outcomes in which there are *no* duplications, that is, permutations of the n "birthdays" chosen r at a time. Therefore, the event, E, consists of ${}_nP_r$ outcomes. Thus, the probability of no duplications is given by

$$\Pr[E] = \Pr[\text{unique birthdays}] = \frac{{}_nP_r}{n^r},$$

and the probability of at least one duplication is given by,

$$\Pr[E'] = 1 - \frac{{}_nP_r}{n^r}.$$

Example 1.2-6 Birthday Problem Probability

Calculate the probability that in a class of 25 students at least one pair of students will have the same birthday.

Solution
From the second formula, the answer is

$$\Pr\left[\text{at least one shared birthday}\right] = 1 - \frac{{}_{365}P_{25}}{365^{25}} = 0.57.$$ ❐

Exercise 1-15 Find the smallest number of students for which the probability of a birthday duplication is at least ½.

Exercise 1-16 What is the smallest number of people in a room to assure that the probability that at least two were born on the same day of the week is at least 40%?

Exercise 1-17 Seven people enter the elevator on the first floor of a 12 story building. What is the probability that no two will get off on the same floor? You may assume that all floors are equally likely and no one gets off the elevator before it starts up.

Exercise 1-18 The church of Individual Self-Actualization has a hymnal consisting of 35 songs. Twelve members come in and open their hymnals to a random song and commence singing (all songs are equally likely). What is the probability that no two congregants are on the same song?

Exercise 1-19 Each individual in a group of n students is asked to pick an integer at random between 1 and 10 (inclusive). What is the smallest value of n that assures at least a 50% chance that at least two students select the same integer?

1.2.5 COMBINATIONS

We now turn to situations in which the *order* of selection doesn't matter. It only matters if an object is selected or not selected. Think of a poker hand. It does not matter the order that you are dealt all four aces and a fifth card; what matters is that you were dealt all of the aces. (Advice: Bet big, this happens infrequently).

Combinations

Given a set of n distinguishable objects, an unordered selection of r different elements of the set is called a ***combination of*** n ***objects chosen*** r ***at a time*** and is denoted by ${}_nC_r$ and read as n choose r.

Combinations apply when order does **not** matter. In determining whether to use combinations or permutations, it is helpful to ask, "Is this like the outcomes of a horserace where order matters?", or "Is this like choosing pizza toppings where order does not matter?"

Example 1.2-7 Combinations

Papa Your Mommas Pizza Parlor has the following four pizza toppings: pepperoni, mushroom, sausage, and olives.

(a) How many different pizzas with two distinct toppings could be ordered?

(b) List the sample space for pizzas with two distinct toppings.

(c) If all two-topping pizzas are equally likely to be ordered, find the probability that sausage is on the pizza.

Solution

The order in which *Papa* places toppings on your pie does not matter. That is, unless you are hopelessly obsessive-compulsive (same as compulsive-obsessive), a pepperoni and mushroom pizza is identical to a mushroom and pepperoni pizza.

(a) Using permutations, we calculate the number of distinct pizzas as ${}_4P_2$ and then divide by the 2! possible orderings of the 2 toppings. Thus, ${}_4C_2 = \frac{4\cdot 3}{2\cdot 1} = 6$.

(b) The six pizzas are pepperoni-mushroom, pepperoni-sausage, pepperoni-olive, mushroom-sausage, mushroom-olive, and sausage-olive. Shorthand notation for our sample space is $\text{Sample Space} = \{PM, PS, PO, MS, MO, SO\}$.

(c) Three pizzas have sausage: pepperoni-sausage, mushroom-sausage, and sausage-olive. Therefore $\Pr(\text{sausage on the 2 topping pizza}) = \frac{3}{6} = 0.5$. ❐

The formula for the number ${}_nC_r$ of combinations can be deduced from the multiplication principle, applied to a *two*-step process for listing the ${}_nP_r$ ordered outcomes.

(1) Choose r objects without regard to order (${}_nC_r$ ways),

(2) Choose a particular ordering (permutation) of the r objects (${}_rP_r = r!$ ways).

Then, ${}_nP_r$ is the product resulting from steps (1) and (2). That is, ${}_nP_r = \frac{n!}{(n-r)!} = {}_nC_r \times r!$. Solving for ${}_nC_r$ leads to the following formula.

Combination Formula

The number of combinations of r objects chosen from a collection of n distinguishable objects is given by,

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!} = \frac{\overbrace{n\cdot(n-1)\cdot(n-2)\cdots(n-r+1)}^{\text{first } r \text{ factors of } n! \text{ on top}}}{\underbrace{r\cdot(r-1)\cdot(r-2)\cdots 3\cdot 2\cdot 1}_{\text{all } r \text{ factors of } r! \text{ on bottom}}} $$

Note: Other common ways to denote combinations: ${}_nC_r = \binom{n}{r} = C(n,r) = C_r^n$

The form ${}_nC_r = \binom{n}{r}$ is especially common and is referred to as the ***binomial coefficient***. The reason for this will be explained in Subsection 1.4.1.

Exercise 1-20 Find the numerical values of the following combinations:

(a) ${}_7C_6$	(b) ${}_{13}C_{12}$	(c) ${}_{13}C_1$
(d) ${}_5C_2$	(e) ${}_4C_4$	(f) ${}_2C_3$
(g) ${}_5C_3$	(h) ${}_4C_0$	(i) ${}_5C_5$
(j) ${}_{5867}C_1$	(k) $\frac{{}_7P_4}{4!}$	(l) ${}_0C_0$

Exercise 1-21 Verify that ${}_{12}C_3 = {}_{12}C_9$.

Exercise 1-22 Verify the relationship ${}_nC_r = {}_nC_{n-r}$. In terms of pizza toppings, explain this relationship.

Exercise 1-23 A father of 6 children comes home to a disheveled house. His slovenly children are to blame. Since he does not know who is responsible, he will select four of the six children for clean-up duty. In how many ways can he do this?

Note: ${}_6C_4 = 15 = {}_6C_2$, since selecting four children to clean up is equivalent to selecting two children to spare from chores.

Exercise 1-24 *Papa Your Mommas Pizza Parlor* has 6 meat toppings and 7 vegetable toppings from which to select. The parlor has three different sizes of pizza (individual, large, and giant) and two different types of crust (deep-dish and thin).

(a) How many different two-topping pizzas could be ordered?

(b) How many different two-topping pizzas could be ordered with exactly one meat topping and exactly one vegetable topping?

(c) How many different four-topping vegetarian pizzas could be ordered?

Exercise 1-25 Coach Cramer has 15 basketball players: four centers, five forwards, and six guards.

(a) How many ways can she select five players to form her starting line-up, regardless of position?

(b) How many ways can she select five players to form her starting line-up if she needs one center, two forwards, and two guards?

Exercise 1-26 Your local Ice Cream Shoppe has eleven flavors of ice cream, three types of cones (waffle, chocolate, and sugar), and five types of toppings (broken peanuts, whipped cream, hot fudge, maraschino cherries, and sprinkles). How many different two-scoop cones can be ordered that have two different flavors of ice cream, exactly one topping, and your choice of cone?

Exercise 1-27 A young man has a collection of fourteen earrings, three nose rings, seven necklaces, and five hemp bracelets. How many different groupings of jewelry may he wear if he selects three earrings, one nose ring, two necklaces, and one bracelet?

Exercise 1-28 A high school lottery uses two sets of numbered balls. One set consists of ten white balls numbered 1-10 and the second set contains twenty blue balls numbered 1-20. To play, you select two white balls and two blue balls.

(a) How many different outcomes are possible?

(b) Your lottery ticket consists of four numbers: two white numbers, each between 1 and 10 inclusive, and two blue numbers, each between 1 and 20, inclusive. What is the probability that your lottery ticket contains exactly one matching white number and two matching blue numbers?

Exercise 1-29 At a picnic, there was a bowl of chocolate candy that had 10 pieces each of Milky Way®, Almond Joy®, Butterfinger®, Nestle Crunch®, Snickers®, and Kit Kat®. Jen grabbed six pieces at random from this bowl of 60 chocolate candies.

(a) What is the probability that she got one of each variety?

(b) What is the probability that Jen grabs exactly five varieties?

1.2.6 PARTITIONS

Whenever we select a combination of r objects from n distinguishable objects, we are in effect dividing the n objects into two classes, those that are "in" and those that are "out." For example, in selecting 5 starters from 15 players the coach probably just reads off the names of the 5 who are "in." Alternatively though, she could convey the same information by reading off the names of the 10 "out" players, telling them to go sit on the bench (admittedly, an approach that emphasizes the negative and probably wouldn't do much for morale.)

Selecting a combination of objects in effect *partitions* the n objects in two sets, one of size r (the *in* set), the other of size $n-r$ (the *out* set). This is another explanation for the relationship ${}_nC_r = {}_nC_{n-r}$ (see Exercise 1-22).

Many combinatorial problems involve partitioning the population (the n distinguishable objects) into *more* than two subsets. The sizes of the subsets are always prescribed in advance.

Partitions

Let A be a set consisting of n distinguishable objects. Let whole numbers $\{r_1, r_2, \ldots, r_k\}$ be given such that $r_1 + r_2 + \cdots + r_k = n.$ A ***partition*** *of A into subsets of sizes* $\{r_1, r_2, \ldots, r_k\}$ is a particular distribution of the n objects into disjoint subsets $A_1, A_2, \ldots, A_k$ of sizes $r_1, r_2, \ldots, r_k$ respectively.

Notes

(1) Since the A_i are disjoint and $r_1 + r_2 + \cdots + r_k = n$, it follows that $A = \bigcup_{i=1}^{k} A_i$.

(2) Once objects are assigned to one of the subsets, their order *within* the subset is irrelevant.

(3) As we saw above, a *combination* of r objects is a partition of A into subsets of size r (the *in* set) and size $n - r$ (the *out* set).

Example 1.2-8 Partitions

An intramural basketball team has five highly versatile starters (Amy, Brandy, Chrystal, Diane, and Erin) each capable of playing any one of the three positions of center, forward, and guard. How many different starting *lineups* consisting of one center, two forwards and two guards are possible? We do not distinguish here between *power forward* and *small forward*, or *point guard* and *shooting guard*.

Solution

A *lineup* consists of one center, two forwards and two guards. Thus, the solution consists of calculating the number of partitions of 5 objects (the starters) into 3 subsets (center, forwards, guards) of sizes 1, 2, and 2 respectively. Any of the five players could play at center, so there are ${}_5C_1 = 5$ choices for center. Of the four remaining players, we must choose two to play at forward – and there are ${}_4C_2 = 6$ ways to accomplish this. The remaining two players must be assigned to play guard – there are ${}_2C_2 = 1$ ways to do this (the one way is to have both remaining ladies play as guards). By the multiplication principle, the total number of line-ups is ${}_5C_1 \times {}_4C_2 \times {}_2C_2 = 5 \times 6 \times 1 = 30$.

Since the numbers are manageable, we can try listing all the partitions.

Position	Center	Forwards	Guards
Size	1	2	2
Set	#1	#2	#3
Lineup (1)	Amy	Brandy, Chrystal	Diane, Erin
Lineup (2)	Amy	Brandy, Diane	Chrystal, Erin
Lineup (3)	Amy	Brandy, Erin	Chrystal, Diane
Lineup (4)	Amy	Chrystal, Diane	Brandy, Erin
⋮	⋮	⋮	⋮
Lineup (30)	Erin	Chrystal, Diane	Amy, Brandy

The enterprising student should complete the list of all 30 partitions. The partial listing above is in alphabetical order. Note that, since the ordering of the forwards and guards doesn't matter, each pair is listed in alphabetical order within position for definiteness. ❒

Systematically listing all possible outcomes is an extremely useful tool. We have already used systematic listing to find the sample spaces in a variety of instances, including: families with three children, flipping coins until a head occurs, rolling a pair of fair dice, constructing wardrobes, and constructing basketball line-ups.

Example 1.2-9 Several Partitions of a Deck of Cards

Suppose the set under consideration is a ***standard deck*** of playing cards, consisting of 52 distinguishable objects (no jokers, missing cards, or wild cards). Dividing the deck into four ***suits*** (clubs, diamonds, hearts, and spades) is a particular partition into 4 subsets, all of size 13.

In the card game ***bridge***, the deck is dealt (divided randomly) into 4 hands (sets, often labeled North, East, South and West) each of size 13. Thus, a bridge hand is one of many possible partitions of the deck into 4 subsets of size 13.

Another special partition of a standard deck would be into the thirteen **denominations, or ranks** of cards (ace, king, queen, …, 2), a partition into 13 subsets, each of size 4.

Partitions of cards need not be of the same size – like in Uno®, gin rummy, and Go Fish. If all 52 cards are dealt to three players, the sets might be partitioned into three hands of size 18, 17, and 17. ❐

Example 1.2-10 Partitions

How many different bridge games are possible?
In other words, how many distinct partitions of 52 objects into 4 subsets of size 13 are possible?

Sample calculations of this type will lead us to a general formula for counting the number of possible partitions. We can break the problem down into a four step process and employ the multiplication principle.

Solution

Step 1: How many ways to choose the cards for North?

Answer: $_{52}C_{13} = \dfrac{52!}{(13!)(39!)} = 635{,}013{,}559{,}600.$

Step 2: How many ways to choose the cards for South?
Answer (since there are just 39 cards left for South):

$$_{39}C_{13} = \frac{39!}{(13!)(26!)} = 8{,}122{,}425{,}444\ .$$

Step 3: How many ways to choose the cards for East?

Answer: $_{26}C_{13} = \dfrac{26!}{(13!)(13!)} = 10{,}400{,}600.$

Step 4: How many ways to choose the cards for West?
Answer (note that after Steps 1-3 there are only 13 cards left):

$$_{13}C_{13} = \frac{13!}{(13!)(0!)} = 1.$$

Thus, the total number of bridge hands is given by

$$\begin{aligned}{}_{52}C_{13}\cdot{}_{39}C_{13}\cdot{}_{26}C_{13}\cdot{}_{13}C_{13} &= \frac{52!}{(13!)(39!)}\cdot\frac{39!}{(13!)(26!)}\cdot\frac{26!}{(13!)(13!)}\cdot\frac{13!}{(13!)(0!)}\\ &= \frac{52!}{(13!)(13!)(13!)(13!)}\\ &= (635{,}013{,}559{,}600)\cdot(8{,}122{,}425{,}444)\cdot(10{,}400{,}600)\cdot(1)\\ &= 53{,}644{,}737{,}765{,}488{,}792{,}839{,}237{,}440{,}000,\end{aligned}$$

which is one heck of a lot of bridge hands. ❐

Example 1.2-11 Partitions of Students

There are a total of nine available seats in three different sections of MATH101. MATH101 section A (MATH101-A) has 3 available seats, MATH101-B has 2 available seats, and MATH101-C has 4 available seats. There are 9 students awaiting assignment to the remaining 9 open slots in MATH101. How many different assignments of students to sections are possible?

Solution
Since the order of the students selected for a given section does not matter, we can view this as the number of ways of partitioning 9 distinguishable objects (students) into sets of size 3, 2, and 4. Thus,

Total ways equals ${}_9C_3\times{}_6C_2\times{}_4C_4=\frac{9!}{(3!)(6!)}\times\frac{6!}{(4!)(2!)}\times\frac{4!}{(4!)(0!)}=\frac{9!}{(3!)(2!)(4!)}=1260.$ ❐

In general the solution to a partitioning problem is called a ***multinomial coefficient***. The examples above can be generalized to the multinomial formula:

Multinomial Coefficients

The number of partitions of n distinct objects into k subsets of sizes $r_1, r_2, \cdots, r_k$, where $r_1+r_2+\cdots+r_k = n$ is called a ***multinomial coefficient***, denoted by $\binom{n}{r_1, r_2, \cdots, r_k}$. The resulting formula is:

$$\binom{n}{r_1, r_2, \cdots, r_k} = \frac{n!}{r_1!\,r_2!\cdots r_k!}.$$

Proof
First, using the multiplication principle as in the preceding examples, we have,

$$\binom{n}{r_1, r_2, \cdots, r_k} = \binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3}\cdots\binom{r_k}{r_k}.$$

Thus,

$$\binom{n}{r_1, r_2, \cdots, r_k} = \left(\frac{n!}{r_1!(n-r_1)!}\right)\left(\frac{(n-r_1)!}{r_2!(n-r_1-r_2)!}\right)\left(\frac{(n-r_1-r_2)!}{r_3!(n-r_1-r_2-r_3)!}\right)$$

$$\cdots\left(\frac{(\overbrace{n-r_1-\cdots-r_{k-1}}^{\text{this equals } r_k})!}{r_k!0!}\right)$$

$$= \left(\frac{n!}{r_1!\cancel{(n-r_1)!}}\right)\left(\frac{\cancel{(n-r_1)!}}{r_2!\cancel{(n-r_1-r_2)}}\right)\left(\frac{\cancel{(n-r_1-r_2!)}}{r_3!(n-r_1-r_2-r_3)!}\right)\cdots\left(\frac{r_k!}{r_k!\cdot 0!}\right)$$

$$= \frac{n!}{r_1!r_2!r_3!\cdots r_k!},$$ where the first several cancellations are noted above.

Example 1.2-12 Multinomial Coefficients

The women's basketball team consists of 15 highly versatile players, each capable of playing all positions. How many different lineups consisting of one center, two forwards, and two guards are possible? **Note:** Chrystal playing guard results in a different lineup from Chrystal playing forward.

Solution 1

We can break this up into a two step process in which the first step is to choose the five starting players. The second step is to partition the five players into subsets of sizes 1, 2, and 2. Using the multiplication principle, the number of different lineups is

$$\underbrace{\binom{15}{5}}_{\substack{\text{ways to}\\ \text{select 5 starters}}} \times \underbrace{\binom{5}{1,2,2}}_{\substack{\text{among 5 starters,}\\ \text{ways to assign positions}}} = \frac{15!}{5!10!}\times\frac{5!}{1!2!2!} = (3003)(30) = 90,090.$$

Solution 2

Alternatively, we can select one student to start at center. There are ${}_{15}C_1 = \frac{15!}{14!\cdot 1!} = 15$ ways to do this. Then we select two of the remaining 14 players to play guard, ${}_{14}C_2 = \frac{14!}{12!\cdot 2!} = 91$ and two women to play forward, ${}_{12}C_2 = \frac{12!}{10!\cdot 2!} = 66$. By the multiplication principle, there are a total of ${}_{15}C_1 \cdot {}_{14}C_2 \cdot {}_{12}C_2 = 90,090$ ways.

Solution 3

Using the multinomial coefficient directly, we have that 15 players need to be partitioned into a group of 1 starting center, 2 starting guards, 2 starting forwards, and 10 players starting on the bench. There are

$$\binom{15}{1,2,2,10} = \frac{15!}{1!\cdot 2!\cdot 2!\cdot 10!} = \frac{15\cdot 14\cdot 13\cdot 12\cdot 11}{4} = 90,090$$

ways to do this. ❒

Note

Does the answer change if the coach selects two starting guards first, then 10 bench players, then two forwards, and finally the center?

Exercise 1-30 Coach Cramer has 15 basketball players – four centers, 5 forwards, and 6 guards. She starts one center, two guards, and two forwards. How many different groups of bench-warmers are possible?

Exercise 1-31 Due to budgetary constraints, Coach Cramer has just eight uniforms for her 15 girls. Conference rules require all players wear a uniform. In how many ways can the coach select five ladies to start, three ladies to dress as subs, and seven ladies to remain in street clothes (she no longer cares about filling guard/forward/center, just ways to start/sub/not dress)?

Using multinomial coefficients allows us to easily calculate the number of different arrangements possible when some of the objects are indistinguishable. For example, we might have a row of colored light-bulbs on a score board consisting of several indistinguishable red bulbs, several more indistinguishable blue bulbs, and so forth. The question would be, "How many distinct arrangements of the colored bulbs are possible?"

Example 1.2-13 Multinomial Coefficients and Different Words

How many different "words" of length 11 are possible using 4 I's, 1 M, 2 P's, and 4 S's?

Solution

The technique is to think of the eleven distinguishable objects in slots forming our "word."

I	I	I	I	M	P	P	S	S	S	S
1	2	3	4	5	6	7	8	9	10	11

Then the question reduces to partitioning the 11 slots into subsets of sizes 4,1,2,4, where the subsets are labeled (and populated) using the letters I, M, P, and S. One such partition is:

Subset Label	I	M	P	S
Subset Size	4	1	2	4
Slots Assigned	2,5,8,11	1	9,10	3,4,6,7

resulting in the particular partition,

Label	M	I	S	S	I	S	S	I	P	P	I
Slot	1	2	3	4	5	6	7	8	9	10	11

Thus, another formulation of the problem is, "how many different arrangements are possible using the letters MISSISSIPPI?"

Answer: $$\binom{11}{4,1,2,4} = \frac{11!}{4!1!2!4!} = 34,650.$$ ❐

Exercise 1-32 How many "words" can be spelled using two O's, an H and a P? What is the probability of choosing the word HOOP assuming arrangements are equally likely?

Exercise 1-33 How many words can be spelled using all of the letters in ABCBA? List them.

Exercise 1-34 How many words can be spelled using all of the letters in *Pennsylvania*?

Exercise 1-35 Six friends are playing poker. Each person is dealt five cards. In how many different ways can this be accomplished? That is, how many ways can five cards each be dealt to six different players in the initial deal of cards? Write your solution in terms of (a) the multiplication principle and (b) multinomial coefficients.

Exercise 1-36 Twenty-five girls are bored.

(a) In how many ways may five girls be chosen to go to a party and five different girls be chosen to volunteer at a soup kitchen (and the remaining 15 girls stay home)?

(b) How many ways may the girls form two teams of 5 to play basketball against each other?

Exercise 1-37 A fraternity consisting of 30 members wants to play "seven versus seven" flag football. How many different match-ups are possible?

Exercise 1-38 Nine workers are assigned to nine jobs. Two of the jobs are considered bad, four are considered average, and three of the jobs are considered good. The nine workers consist of seven men and two women. If workers are randomly assigned to jobs, calculate the probability that the two women are both assigned to bad jobs.

1.3 SAMPLING AND DISTRIBUTION

The best way to master combinatorial (counting) type problems is to completely understand a few of the basic building-blocks, and to learn how seemingly unrelated problems are in fact structurally the same. In this section we provide a structure to formally classify certain combinatorial problems by their particular type.

One common formulation for these types of problems – as we have seen in examples from earlier sections – involves choosing subsets from a set of n distinguishable objects. In statistics, these subsets (or events) are called ***samples***. A different way of formulating problems involves assigning, or ***distributing****,* markers to the n objects. It turns out that these two types of problems, *sampling* and *distribution*, are closely related. In fact every sampling problem can be recast as a distribution problem, and vice-versa.

Mathematicians often couch sampling problems in terms of *removing balls from urns*. The related assignment problem would be posed as *distributing balls into urns*. In this sense, the distribution of balls into urns is frequently referred to as *occupancy*. We have never understood the fascination with balls and urns, but the terminology pervades in the classical textbooks and is convenient to use in designing examples.

1.3.1 SAMPLING

We begin with sampling problems. Consider a set consisting of n distinguishable objects (for example, numbered balls in an urn). The set of n objects is called the *population*. Assuming all outcomes are equally likely, the probability model boils down to the question "How many distinct samples of size r can be drawn from the n objects?"

Before we can answer this big question we need the answers to two related questions:

I. Are the samples taken with or without replacement (that is, can we pick an object at most once, or can we pick the same object more than once)?

II. Does the order in which we select the items in the sample matter?

Example 1.3-1 The Four Different Types of Sampling

For each of the four types of sampling described below, calculate the number of distinct outcomes.

(1) A club consisting of 26 members needs to select an executive board consisting of a president, a vice-president, a secretary and a treasurer. Each position has different duties and responsibilities. No individual can hold more than one office.

(2) A club consisting of 26 members needs to select a delegation of 4 (distinct) members to attend a convention. The delegation of 4 wears identical goofy hats.

(3) A club consisting of 26 members requires volunteers to complete 4 distinct chores; sweeping the clubhouse, printing off raffle tickets for the drawing at the party that night, picking up the prizes, and picking up the empties after the party. The same person can volunteer for more than one job, and each job requires only one volunteer.

(4) A club consisting of 26 members requires volunteers to make 4 identical recruiting phone calls, to be chosen later from a long list. A member can volunteer for one, two, three, or all four calls.

We classify these four situations with respect to replacement/without replacement, and order matters/order doesn't matter.

Example	***n***	***r***	**Replacement?**	**Order Matters?**
#1	26	4	No	Yes
#2	26	4	No	No
#3	26	4	Yes	Yes
#4	26	4	Yes	No

Or,

	Without replacement	**With replacement**
Order matters	Example 1	Example 3
Order doesn't matter	Example 2	Example 4

In (1) and (2) the context makes clear we want to select four different people, so in both cases we are sampling without replacement. In (1) the order matters since if Rajulio is selected it makes a difference as to which office he holds. In (2), order doesn't matter. We are only concerned with the members of the delegation, not the order in which they were selected. In (3) and (4), since members can volunteer for multiple chores, we are sampling with replacement. In (3) the chores are different, so we need to keep track of who is volunteering for which job. In (4) the calls are essentially identical, so we are only concerned with how many calls a volunteer makes.

Each member of the club can be uniquely identified (conveniently, with a single letter of the alphabet since there are exactly 26 members). The outcomes for each of these four sampling experiments can be put into one-to-one correspondence with certain types of four letter "words." "Words" (in quotes) means, as previously, any list of four letters, not just dictionary words.

In sampling without replacement, no letter can be repeated in a word. If order matters then ABCD is different from DCBA. If order does not matter, then we agree to list the four letters chosen for the sample just once, in alphabetical order. Thus, we can rephrase our four questions this way:

(1) Calculate the number of four letter words with no duplication of letters.

(2) Calculate the number of four letter words with no duplicate letters and the letters arranged in alphabetical order.

(3) Calculate the number of four letter words, duplicate letters allowed.

(4) Calculate the number of four letter words, duplicate letters allowed, and the letters arranged in alphabetical order.

Solution

The first three are straightforward to work out using permutations, combinations and the basic multiplication principle, respectively.

(1) Answer: $\underset{\text{president}}{\underline{26}} \cdot \underset{\text{vice-pres}}{\underline{25}} \cdot \underset{\text{secretary}}{\underline{24}} \cdot \underset{\text{treasurer}}{\underline{23}} = {}_{26}P_4 = 358{,}800.$

(2) Answer: ${}_{26}C_4 = \dfrac{{}_{26}P_4}{4!} = \dbinom{26}{4} = 14{,}950.$

(3) Answer: $\underset{\text{sweep}}{\underline{26}} \cdot \underset{\text{raffle}}{\underline{26}} \cdot \underset{\text{prizes}}{\underline{26}} \cdot \underset{\text{empties}}{\underline{26}} = 26^4 = 456{,}976.$

Example (4) is a little more subtle and we will defer the solution to the next section where it will be more easily understood as a distribution problem. ❐

1.3.2 DISTRIBUTIONS

Each of the four types of sampling experiments described above has a dual formulation as a *distribution*, or occupancy, experiment. For this purpose, we let n be the number of distinguishable (fixed and labeled) urns, and let r be the number of balls to be distributed into the urns. How many different distributions are possible? Again, before we can answer the question, we need to know:

I. Can an urn hold at most one ball (exclusive) or can it hold many balls (non-exclusive)?

II. Are the balls distinguishable (for example, bearing unique numbers) or are they indistinguishable (like plain white ping-pong balls)?

We can rephrase each of the four sampling problems above as a corresponding distribution problem:

(1) In how many ways can 4 executive board positions (distinguishable balls) be distributed among 26 members (urns) with exclusion (since no member can hold more than one position)?

(2) In how many ways can 4 delegation slots (indistinguishable balls) be distributed among 26 members (urns) with exclusion?

(3) In how many ways can 4 different jobs (distinguishable balls) be distributed among 26 members (urns) without exclusion (since one member can do multiple jobs)?

(4) In how many ways can 4 identical jobs (indistinguishable balls) be distributed among 26 members (urns) without exclusion (since one member can do multiple jobs)?

Statements (1)-(3) were solved in Section 1.3.1 formulated as equivalent sampling problems. Here is a paradigm for counting in sampling or distribution problems like (4).

Imagine the 26 urns lined up in a row sharing common partitions:

A	*B*	*C*	$\cdots$	*Z*

We now distribute the 4 indistinguishable balls among the 26 urns. A sample outcome might be:

$\circ$	$\circ\circ$			$\circ$
A	*B*	*C*	$\cdots$	*Z*

Now, observe that the top row in the above schematic can be thought of as a word consisting of (26 + 1) lines and 4 circles. However, the leftmost line and the rightmost line remain fixed in every word and are therefore superfluous. A moment's thought will show that there is a one-to-one correspondence between distinct distributions into the 26 slots, and distinct words consisting of $(26-1)$ lines and 4 circles. The distribution above appears as

$$\circ|\circ\circ|||\cdots|\circ.$$

Thus, the question is reduced to, "How many $(26-1+4)$ letter words are there consisting of four circles and $(26-1)$ vertical lines?" Therefore, the solution to example (4) is

$$\binom{26-1+4}{4} = \binom{29}{4} = 23{,}751.$$

The above discussion leads us to the following result:

Samples with Replacement When Order Does Not Matter

The number of unordered samples of *r* objects, with replacement, from *n* distinguishable objects is

$${}_{n+r-1}C_r = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}.$$

This is equivalent to the number of ways to distribute *r* indistinguishable balls into *n* distinguishable urns without exclusion.

The same reasoning works in general, so that ${}_{n+r-1}C_r$ is the solution to the following sampling and distribution problems:

Sampling: How many distinct unordered samples of size *r*, with replacement, are there from n distinguishable objects?

Distribution: How many distinct ways are there to distribute *r* indistinguishable balls into n distinguishable urns, with multiple balls in an urn allowed?

We like to think of these types of problems in terms of distributing r dollar bills to n children. One needs $n-1$ partitions to separate the children, and then select the locations to place the r dollars.

Example 1.3-2 Samples with Replacement When Order Does Not Matter

Nicole wishes to select a dozen bagels from *Unleavened Bread Company*. Her choices include: Asiago cheese, plain, nine grain, cinnamon crunch, and very-very blueberry.

(a) How many different orders of a dozen bagels can she select?

(b) How many different orders of a dozen bagels can she select in which she has at least one of each kind?

Solution

Selecting a dozen bagels is equivalent to distributing 12 indistinguishable markers (ordering a bagel) into 5 (bagel) bins. Our answer to (a) is

$$_{(5-1+12)}C_{12} = {}_{16}C_{12} = 1{,}820.$$

For part (b), imagine that we first select one of each type of bagel. Our problem reduces to selecting the remaining seven bagels from any the five types. This can be done in

$$_{(5-1+7)}C_7 = {}_{11}C_7 = 330 \text{ ways.}$$ ❐

The "Boston Chicken" example, below, illustrates how identifying the particular type of sample can be very important. Boston Chicken has 16 distinct side-dishes (the population). Each dinner is served with the customer's choice of three side-dishes. Read the following article and explain the advertisement that there were "more than 3000 combinations" possible.

Thurs, Jan. 26, 1995. Rocky Mountain News, Retail & Marketing section

Catching Boston Chicken's error made Bob Swaim king for a week

Ad helps teacher prove math's important

By Lynn Bronikowski
Rocky Mountain News Staff Writer

Nearly every day, Pennsylvania high school teacher Bob Swaim harps about the importance of math skills.

Some students yawn; others chow down on chicken — the reward for Swaim's attention to detail in the world of television advertising.

The Golden-based Boston Chicken rotisserie chicken restaurant chain earlier this month aired an ad for its combo platter in which it said a customer could get more than 3,000 combinations by ordering a three-item combo.

But Swaim, who has taught math for 28 years, caught Boston Chicken with egg on its face after pointing out there are actually only 816 *combinations* but more than 3,000 *permutations.*

Boston Chicken, which corrected the ad by last weekend, won a flood of publicity by admitting the goof. Stories appeared locally and in *USA Today, The Toronto Star, Orange County Register, Chicago Sun Times, Cincinnati Post* and the Associated Press.

National Public Radio picked up the story and producers from TV network shows made calls.

Swaim, 48, became a small-town hero in Souderton, Pa. Boston Chicken donated $500 to the math department and treated 30 trigonometry students to lunch at a nearby Boston Chicken restaurant last Friday.

Swaim never actually saw the commercial starring Joe Montana, giving credit to his 15-year-old daughter, Joann, who first spotted the miscalculation.

"Everyday I'm telling my students that math is important," said Swaim, who teaches two trig classes, a geometry class and math for daily living. "I felt I had a moral responsibility to call about the mistake."

Swaim said at first he got a runaround before finally talking to the copywriter at the Integer Group in Golden.

He was surprised no one else had called. "Their phones should have been ringing off the hook," said Swaim. "There should have been more people than just me (questioning) it.

"It points out some of the weaknesses in our culture. It was such a simple problem and I'm no whiz kid."

He figures a second-year algebra student could have worked the formula.

"We're not used to this happening; Souderton is such a small town," said Jacey Stroback, a junior in one of Swaim's two trig classes. "He's always talking about how we're going to use math skills later on in life, no matter what you do."

Even before the math encounter, Swaim was a Boston Chicken fan, having eaten the restaurant's food at home. And after spotting a flier in a nearby restaurant, he invited a speaker to talk to two classes about career opportunities.

"I'm always looking for people to promote positive education and I liked their attitude," said Swaim.

After his fleeting moment of fame, life has returned to normal. But not before the producers of Tom Snyder's late-night talk show called about an appearance.

"I got bumped by a guy who collects stray birds."

Permutated veggies: a reworked recipe

Boston Chicken made its mistake by confusing combinations and permutations. One combination can count as several permutations.

Corn-corn-mashed potato and corn-mashed potato-corn are two different *permutations*, but you still wind up with the same *combination* of foods.

The chain provided this corrected formula for calculating the number of possible three-item samplers that can be made from 16 Boston Chicken side items.

Add up all of the following scenarios:

- Selecting all three of the same items. For example, corn-corn-corn. Equals 16 ways.
- Selecting two of the same items and a third different item. For example, corn-corn-mashed potato. The first one in 16 ways; second one in 15 ways; third one in two ways (as you selected just one of the two already selected); remove combinations that are the same. Equals 240 ways.
- Selecting all three different items. For example, corn-stuffing-mashed potato. First one in 16 ways; second one in 15 ways; third one in 14 ways; remove combinations that are the same. Equals 560 ways.
- Total all of the above number of ways. Equals 816 ways.

Exercise 1-39 Calculate the number of possibilities under ordered samples without replacement (customers eat their side dishes in the order of selection and must choose 3 different dishes).

Exercise 1-40 Calculate the number of possibilities using unordered samples without replacement (customers can eat their dishes in any order but still must choose 3 different dishes).

Exercise 1-41 Calculate the number of possibilities using ordered samples with replacement (customers eat their side dishes in a definite order, but can order, for example, corn-corn-mashed potato, which is different from corn-mashed potato-corn.

Exercise 1.42 Calculate the number of possibilities using unordered samples with replacement (customers eat their side dishes in any order, and corn-corn-mashed potato is possible and is identical to corn-mashed potato-corn.

Exercise 1-43 Which type of sampling gives an answer closest to “more than 3000?”

Exercise 1-44 Which type of sampling provides the most realistic result?
Hint: The correct answer of 816 is given in the side-bar of the article, although it is worked out differently using multiple steps.

Exercise 1-45 Assume that Boston Chicken customers choose three side-dishes from the 16 possible in such a way that all unordered samples with replacement are equally likely. What is the probability that a customer will choose all three side-dishes the same? That is, in poker parlance, what is the probability of three-of-a-kind?

Exercise 1-46
(a) How many ways can a parent distribute five one-dollar bills to her three children?
(b) How many ways can she accomplish this if each child gets at least one dollar?

Exercise 1-47
(a) How many ways can a witch distribute ten candy bars and seven packages of gum to four trick-or-treaters?
(b) How many ways can she do this if each child receives at least one candy bar and one package of gum?

Exercise 1-48 How many ways may a parent distribute ten identical pickled beets to his five children?

Exercise 1-49 How many different 13-card bridge hands are possible?

Exercise 1-50 How many 5-card poker hands are there?

Exercise 1-51 At a local fast-food restaurant in Oregon (no sales tax), fries, soda, hamburgers, cherry pie, and sundaes cost \$1 each. Chicken sandwiches cost \$2 each. You have five dollars. How many different meals can you order?

Exercise 1-52 I have fifteen certificates for a free pizza and 24 cans of Coca-Cola®. How many ways may I distribute the certificates and the cans of coke to 22 students?

1.3.3 SAMPLING AND OCCUPANCY UNITED

The following diagram shows the complete set of correspondences between sampling and distribution. Although the formulation of a sampling problem may appear to be quite dissimilar to the corresponding distribution problem, the two are in fact mathematically equivalent.

Samples of size *r* from *n* distinguishable objects	**Without replacement**	**With replacement**	
Order matters	$_nP_r$ (Example 1)	n^r (Example 3)	Distinguishable balls
Order doesn't matter	$\binom{n}{r}$ (Example 2)	$\binom{n+r-1}{r}$ (Example 4)	Indistinguishable balls
	Exclusive	Non-exclusive	Distributions of r balls into n distinguishable urns

1.4 MORE APPLICATIONS

In this section we bring together a variety of special applications of the combinatorial methods previously discussed. We begin with an illustration of some basic results involving the algebra of polynomials.

1.4.1 THE BINOMIAL AND MULTINOMIAL THEOREMS

Consider the problem of expanding the binomial expression

$$(x+y)^n = \underbrace{(x+y)}_{\text{factor 1}}\underbrace{(x+y)}_{\text{factor 2}}\underbrace{(x+y)}_{\text{factor 3}}\cdots\underbrace{(x+y)}_{\text{factor } n}.$$

Algebraically, this amounts to adding together all possible products consisting of n letters, the first selected from "factor 1," the second from "factor 2," and so forth up to "factor n." In other words, the expansion consists of the sum of all n letter "words" consisting of the letters x and y.

To simplify our final result, we group together all words that contain the same number of x's and y's. Also, a stickler for algebra would insist on proper exponent notation (and the fact that multiplication is commutative), to express words like *xxyxyyx* and *yxxxyxy* (with 4 x's and 3y's) all as x^4y^3.

Now, for a given r $(r = 0,1,2,\ldots,n)$, the ***number*** of distinct n letter words with (r) x's and $(n-r)$ y's will be the coefficient of $x^r \cdot y^{n-r}$ in the expansion of $(x+y)^n$. As we have previously seen, this is just the combination ${}_nC_r = \binom{n}{r}$. This is the reason that the expression "binomial coefficient" is used to describe the coefficient $\binom{n}{r}$ of the term $x^r \cdot y^{n-r}$ of the binomial expansion for $(x+y)^n$.

The formula in the Binomial Theorem now follows from this observation.

The Binomial Theorem

For every non-negative integer n and real numbers x and y, we have

$$\begin{aligned}(x+y)^n &= \sum_{r=0}^{n} {}_nC_r \cdot x^r \cdot y^{n-r} \\ &= {}_nC_0 \cdot y^n + {}_nC_1 \cdot x^1 \cdot y^{n-1} + {}_nC_2 \cdot x^2 \cdot y^{n-2} + \cdots + {}_nC_n \cdot x^n,\end{aligned}$$

or equivalently,

$$\begin{aligned}(x+y)^n &= \sum_{r=0}^{n} {}_nC_r \cdot x^{n-r} \cdot y^r \\ &= {}_nC_0 \cdot x^n + {}_nC_1 \cdot x^{n-1} \cdot y^1 + {}_nC_2 \cdot x^{n-2} \cdot y^2 + \cdots + {}_nC_n \cdot y^n\end{aligned}$$

Note

The two forms of the theorem given are in fact equivalent since ${}_nC_r = {}_nC_{n-r}$ is the coefficient of both $x^r \cdot y^{n-r}$ and $x^{n-r} \cdot y^r$.

Example 1.4-1 The Binomial Theorem

Use the Binomial Theorem to expand $(x-2y)^3$.

Solution
To apply the binomial theorem, we rewrite $(x-2y)^3 = (x+\{-2y\})^3$.

$$\begin{aligned}(x-2y)^3 &= {}_3C_0 \cdot x^3 \cdot (-2y)^0 + {}_3C_1 \cdot x^2 \cdot (-2y)^1 + {}_3C_2 \cdot x^1 \cdot (-2y)^2 + {}_3C_3 \cdot x^0 \cdot (-2y)^3 \\ &= 1\cdot x^3 \cdot 1 + 3\cdot x^2(-2y) + 3\cdot x\cdot(4y^2) + 1\cdot 1\cdot(-8y^3) = x^3 - 6x^2y + 12xy^2 - 8y^3\end{aligned}$$ ❐

Exercise 1-53 Use the Binomial Theorem to expand the following:

(a) $(x+3)^4$
(b) $(2x+y)^5$
(c) $(4x-5y)^3$

Exercise 1-54 Use the mathematical definition of combinations to verify the identity:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

Note 1
For a *non*-algebraic proof, imagine that you belong to a sorority with n members, exactly r of whom get to go to a party at the fraternity next door. How many ways to select the lucky r members? Break it down into those combinations that contain you, and those combinations that do not contain you. Are you one of the party-goers? If so, it remains to choose $r-1$ members from the $n-1$ members who are not you. If you stay home completing your probability homework, then we must count the ways r members are chosen from the $n-1$ members who are not you.

Note 2
This identity is what makes the generation of Pascal's[3] triangle possible. The n^{th} row of Pascal's triangle contains the binomial coefficients for expanding $(x+y)^n$. Each new row is generated by adding the adjacent coefficients (as in the identity above) from the previous row:

Pascal's Triangle

$(x+y)^0$					1					0^{th} row
$(x+y)^1$				1		1				1^{st} row
$(x+y)^2$			1 ↘		↙ 2		1			2^{nd} row
$(x+y)^3$		1		3		3		1		3^{rd} row
$(x+y)^4$	1		4		6		4		1	4^{th} row

[3] Pascal (1623-1662), considered to be, along with his contemporary, Pierre de Fermat (1601-1665), a progenitor of modern probability theory.

Exercise 1-55 Use the binomial theorem to verify the identity $\sum_{k=0}^{n} {}_nC_k = 2^n$. For example, if $n = 4$, then $\sum_{k=0}^{4} {}_4C_k = {}_4C_0 + {}_4C_1 + {}_4C_2 + {}_4C_3 + {}_4C_4 = 16 = 2^4$.

Note

${}_nC_k$ is the number of $k-$topping pizzas that can be made if there are *n* toppings from which to select. You may create a zero-topping pizza, or a one-topping pizza, up to an *n*-topping pizza. The left-hand side of the equation is the sum of different numbers of *k*-topping pizzas. On the other hand, each pizza topping may either be placed on the pizza or not. There are two ways to do this (put topping on pizza or do not put topping on pizza). There are *n* toppings from which to select. By the multiplication principle, there are $\underbrace{2 \cdot 2 \cdot \cdots \cdot 2}_{n \text{ times}} = 2^n$ ways to do this, which is the right-hand side of the identity.

Exercise 1-56 Explain the fundamental identity $\binom{n}{r} = \binom{n}{n-r}$ in terms of pizza toppings.

Exercise 1-57 In the expansion of $(x+y)^n$ the coefficient of x^4y^{n-4} is 3,876 and the coefficient of x^5y^{n-5} is 11,628. Find the coefficient of x^5y^{n-4} in the expansion of $(x+y)^{n+1}$. What is the value of *n*?

It is possible to generalize the arguments involving the binomial theorem in order to demonstrate the multinomial theorem. We wish to expand out the expression

$$(x_1 + x_2 + \cdots + x_r)^n = \underbrace{(x_1 + x_2 + \cdots + x_r)}_{\text{factor 1}}\underbrace{(x_1 + x_2 + \cdots + x_r)}_{\text{factor 2}}\cdots\underbrace{(x_1 + x_2 + \cdots + x_r)}_{\text{factor } n}.$$

The expansion now consists of all possible n-letter words consisting of the letters $x_1, x_2, \ldots, x_r$. We again group together all words with the same exponents on the letters. How many words are there with (n_1) x_1's, (n_2) x_2's, (n_3) x_3's, …, (n_r) x_r's (think MISSISSIPPI)? The answer is the number of partitions of *n* objects into subsets of sizes $n_1, n_2, n_3, \ldots, n_r$, that is $\binom{n}{n_1, n_2, \cdots, n_r}$. Therefore, the multinomial expansion can be expressed as follows:

The Multinomial Theorem

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{n_1 + \cdots + n_r = n} \binom{n}{n_1, n_2, \cdots, n_r} \cdot x_1^{n_1} \cdot x_2^{n_2} \cdot \cdots \cdot x_r^{n_r}.$$

The sum runs over all possible partitions $n_1, n_2, n_3, \ldots, n_r$ such that $n_1 + n_2 + n_3 + \cdots + n_r = n$. Finding all of these partitions is the challenging part.

Note
The number of such partitions is the number of ways of distributing n indistinguishable balls into r urns non-exclusively, that is, $\binom{n+r-1}{n}$.

Example 1.4-2 The Multinomial Theorem

Use the multinomial theorem to expand $(x+y+z)^4$.

Solution
You should take the time to find all of the ways to combine x, y, and z so that the total powers sum to 4.

$$\begin{aligned}(x+y+z)^4 &= \binom{4}{4,0,0}x^4y^0z^0 + \binom{4}{3,1,0}x^3y^1z^0 + \cdots + \binom{4}{0,0,4}x^0y^0z^4 \\ &= x^4 + 4x^3y + 4x^3z + 6x^2y^2 + 6x^2z^2 + 12x^2yz + 4xy^3 + 12xy^2z \\ &\qquad + 12xyz^2 + 4xz^3 + y^4 + 4y^3z + 6y^2z^2 + 4yz^3 + z^4\end{aligned}$$

Note that the number of terms is 15, the same as $\binom{n+r-1}{n} = \binom{4+3-1}{4}$. ❒

Exercise 1-58 Use the multinomial theorem to expand the following:

(a) $(x-2y+5z)^3$

(b) $(w+x-y+2z)^2$.

1.4.2 POKER HANDS

Our simple version of ***poker*** is played with a standard four-suit, 52-card deck. We are serious players, so there are neither jokers nor wild cards in our deck. A poker hand consists of 5 cards dealt from a standard deck. In other words, a poker hand is an unordered random sample of size 5 chosen from a population of size 52, without replacement (you wouldn't want to be caught in Dodge City with 2 Queens of Hearts in your hand). The ace can be played as either high or low, as explained below. We present the definitions of the various types of poker hands.

Straight flush: Five cards of the same suit in sequence, such as 7♥6♥5♥4♥3♥. The Ace-King-Queen-Jack-Ten (A♣K♣Q♣J♣T♣) is called a ***royal flush***. The ace can also play low so that 5♣4♣3♣2♣A♣ is another straight flush.

Four-of-a-kind: Four cards of the same denomination accompanied by another card, like 7♣7♦7♥7♠9♥.

Full house (*a.k.a*. a ***boat***): Three cards of one denomination accompanied by two of another, such as Q♣Q♦Q♥4♠4♥.

Flush: Five cards of the same suit, such as K♠Q♠9♠6♠4♠. Straight flushes are excluded (they form their own category above).

Straight: Five cards in sequence, such as J♥T♦9♣8♦7♠. The ace plays either high or low, but a collection like 32AKQ is not allowed. That is, there are no wrap-around straights. Again, straight flushes are excluded, since they have been counted separately.

Three-of-a-kind: Three cards of the same denomination and two other cards of different denominations, such as 7♠7♣7♦K♣2♦.

Two Pair: Two cards of one denomination, two cards of another denomination and a fifth card of a third denomination, such as K♠K♣8♦8♠7♥.

One Pair: Two cards of one denomination accompanied by three cards of different denominations, such as T♠T♣Q♦8♠7♥.

High Card (*a.k.a.* Nothing): Any hand that does not qualify as one of the hands above, such as A♠Q♣9♦8♠7♥.

Various combinatorial techniques are employed to calculate the probabilities of being dealt these hands on an initial deal from a standard deck of cards. We illustrate several of these calculations and provide a summary of the probabilities of all types of poker hands.

Since poker hands consist of five cards (any order) selected without replacement from a population of size 52 cards, the size of the sample space is $\binom{52}{5} = 2{,}598{,}960.$

Example 1.4-3 Straight Flush

Compute the probability of being dealt a straight flush.

Solution

The highest ranked straight flush is AKQJT in one of the four suits (clubs, diamonds, hearts, and spades), the lowest ranked straight flush is 5432A. There are 10 of these rankings. There are four suit choices. By the multiplication rule, there are $10 \cdot 4 = 40$ possible straight flushes.

$$\Pr(\text{straight flush}) = \frac{\text{number of straight flushes}}{\text{number of five card poker hands}} = \frac{10 \cdot 4}{\binom{52}{5}} = \frac{40}{2{,}598{,}960} = .00001539. \qquad ❒$$

Example 1.4-4 Full House

Compute the probability of being dealt a full house on the initial deal of 5 cards.

Solution
We begin by selecting a card denomination (Ace, King,..., 2) for the three-of-a-kind. One has 13 ways to do this. We then select three of the four cards from the selected denomination to create the three-of-a-kind. For the pair, there are only 12 remaining denominations to select, since the card value chosen for the three-of-a-kind is no longer available. We need to choose two cards with this second denomination.

$$\Pr(\text{full house}) = \frac{\text{number of full houses}}{\text{number of five card poker hands}} = \frac{\binom{13}{1}\cdot\binom{4}{3}\cdot\binom{12}{1}\cdot\binom{4}{2}}{\binom{52}{5}} = \frac{13\cdot 4\cdot 12\cdot 6}{2{,}598{,}960} = .00144058. \qquad ❒$$

Exercise 1-59 Compute the probabilities for all nine poker hand types. Do this by yourself prior to looking at the following summary. It is an important exercise, even if you struggle.

Poker Probability Summary:

1. **Straight Flush**

$$\Pr(\text{Straight Flush}) = \frac{\text{number of straight flushes}}{\text{number of five card poker hands}} = \frac{4 \text{ suits} \cdot 10 \text{ possible straights}}{\binom{52}{5}} = \frac{40}{2{,}598{,}960} = .00001538.$$

2. **Four-of-a-Kind**

$$\Pr(\text{Four-of-a-Kind}) = \frac{\binom{13}{1}\cdot\binom{4}{4}\cdot\binom{12}{1}\cdot\binom{4}{1}}{\binom{52}{5}} = \frac{13\cdot 1\cdot 12\cdot 4}{2{,}598{,}960} = \frac{624}{2{,}598{,}960} = .00024010.$$

3. **Full House**

$$\Pr(\text{Full House}) = \frac{\binom{13}{1}\cdot\binom{4}{3}\cdot\binom{12}{1}\cdot\binom{4}{2}}{\binom{52}{5}} = \frac{13\cdot 4\cdot 12\cdot 6}{2,598,960} = \frac{3,744}{2,598,960} = .00144058.$$

4. **Flush**

$$\Pr(\text{Flush}) = \frac{\overbrace{\binom{4}{1}}^{\text{suits}}\cdot\overbrace{\binom{13}{5}}^{\substack{\text{5 cards selected}\\ \text{from the suit}}} - \overbrace{40}^{\substack{\text{straight flushes}\\ \text{already accounted}}}}{\binom{52}{5}}$$

$$= \frac{4\cdot 1,287 - 40}{2,598,960} = \frac{5,108}{2,598,960} = .00196540.$$

5. **Straight**

$$\Pr(\text{Straight}) = \frac{10\cdot\binom{4}{1}\cdot\binom{4}{1}\cdot\binom{4}{1}\cdot\binom{4}{1}\cdot\binom{4}{1} - 40}{\binom{52}{5}}$$

$$= \frac{10\cdot 4^5 - 40}{2,598,960} = \frac{10,200}{2,598,960} = .00392465.$$

6. **Three-of-a-Kind**

$$\Pr(\text{Three-of-a-Kind}) = \frac{\binom{13}{1}\cdot\binom{4}{3}\cdot\binom{12}{2}\cdot\binom{4}{1}\cdot\binom{4}{1}}{\binom{52}{5}}$$

$$= \frac{13\cdot 4\cdot 66\cdot 4\cdot 4}{2,598,960} = \frac{54,912}{2,598,960} = .02112846.$$

7. **Two Pair**

$$\Pr(\text{Two Pair}) = \frac{\binom{13}{2}\cdot\binom{4}{2}\cdot\binom{4}{2}\cdot\binom{11}{1}\cdot\binom{4}{1}}{\binom{52}{5}}$$

$$= \frac{78\cdot 6\cdot 6\cdot 11\cdot 4}{2,598,960} = \frac{123,552}{2,598,960} = .04753902.$$

8. **One Pair**

$$\Pr(\text{One Pair}) = \frac{\binom{13}{1}\cdot\binom{4}{2}\cdot\binom{12}{3}\cdot\binom{4}{1}\cdot\binom{4}{1}\cdot\binom{4}{1}}{\binom{52}{5}}$$

$$= \frac{13\cdot 6\cdot 220\cdot 4^3}{2{,}598{,}960} = \frac{1{,}098{,}240}{2{,}598{,}960} = .42256903$$

9. **Nothing**

$$\Pr(\text{Nothing}) = 1 - \Pr(\text{other possibilities})$$

$$= \frac{\binom{52}{5} - (40 + 624 + 3{,}744 + 5{,}108 + 10{,}200 + 54{,}912 + 123{,}552 + 1{,}098{,}240)}{\binom{52}{5}}$$

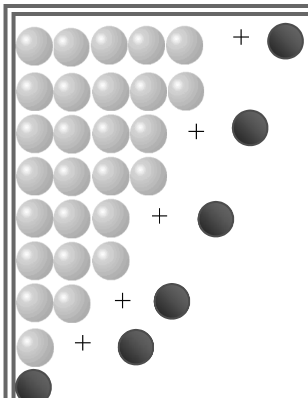
$$= \frac{2{,}598{,}960 - 1{,}296{,}420}{2{,}598{,}960} = \frac{1{,}302{,}540}{2{,}598{,}960} = .50117739.$$

Exercise 1-60 (Poker Dice): Play poker using 5 fair dice rather than a deck of cards. Roll the five dice onto the table. The possible *hands* are: five-of-a-kind, four-of-a-kind, a full house, three-of-a-kind, two pair, one pair, a straight, and nothing. Find the probabilities of these events on a single roll of the dice. Where should the straight ***rank***? The less likely (lower probability) an event is to happen, the higher it should rank.

1.4.3 THE POWERBALL® LOTTERY

The following rules, prizes, and odds were once found at www.musl.com. To play the game, we draw five balls out of a drum with 53 numbered white balls, and one power ball out of a drum with 42 numbered green balls.

Powerball® Prizes and Odds



Match	Prize	Odds
	Grand Prize	1 in 120,526,770.00
	$100,000	1 in 2,939,677.32
	$5,000	1 in 502,194.88
	$100	1 in 12,248.66
	$100	1 in 10,685.00
	$7	1 in 260.61
	$7	1 in 1696.85
	$4	1 in 123.88
	$3	1 in 70.39

The overall odds of winning a prize are 1 in 36.06.
The odds presented here are based on a $1 play and are rounded to two decimal places.

Example 1.4-5 Probability of winning the $5,000 Powerball® lottery prize

Calculate the probability of winning the $5,000 prize (i.e., matching four of the five white balls and the green power ball).

Note
The amount that you win is $4,999 since it costs you a dollar to purchase the game ticket.

Solution
There are five *winning* white balls, of which we need to select four. There are forty-eight $(53-5)$ *losing* white balls, of which we need to select one. We also need to select the correct green power ball, of which there is only one.

$$\begin{aligned}
\Pr(\text{4 white and the power ball}) &= \frac{\binom{5}{4}\cdot\binom{48}{1}\cdot\binom{1}{1}}{\binom{53}{5}\cdot\binom{42}{1}} \\
&= \frac{5\cdot 48\cdot 1}{2{,}869{,}685\cdot 42} \\
&= \frac{240}{120{,}526{,}770} \\
&= \frac{1}{502{,}194.88} \\
&= 0.000001991.
\end{aligned}$$

The Powerball® graphic uses the expression "odds" in the third column. They should have used "probability" and written the expressions as $1/502{,}194.88$. There are a number of ways in which odds are stated, but all of them are variants of probability statements.

In gambling, it is common to give the ratio of expected losses to expected wins in a fixed number of plays. This ratio is referred to as "the odds against winning." For example, suppose that the betting public's favorite horse in the Santa Anita derby has odds of 2:1 (read as "two to one"). This implies that if three identical races were run, then our horse would be expected to *lose* twice and *win* once (the word "expect" is given a precise meaning in Chapter 3). We could translate these odds to the equivalent probability of winning the race, $\Pr(\text{win}) = \frac{\text{1 win}}{\text{1 win+2 losses}} = \frac{1}{3}$. ❒

Consider an event A whose probability of occurring is p. The ***complementary*** event (that A does ***not*** occur) is denoted A'.

Odds

The ***odds*** against the event A are quoted as the ratio,

$$\Pr(A \text{ does not occur}) : \Pr(A \text{ does occur}) = \Pr(A') : \Pr(A) = (1-p) : p.$$

Note: Odds are generally quoted using whole numbers. For example, if $P(A) = \frac{2}{5}$ then the odds against A are $\frac{3}{5} : \frac{2}{5} = 3:2.$

If you wanted to bet on A occurring, and the odds against A are 3:2, then you would put \$2 in the pot and your opponent would put in \$3. If the experiment could be repeated 5 times, then you would expect to win twice and lose three times. You would lose \$6 on the three losses and win \$6 on the two wins, breaking even. This would be considered a ***fair*** bet.

Converting Odds to Probability

If the odds against the event A are quoted as $b:a$, then $\Pr(A) = \dfrac{a}{a+b}$.

Example 1.4-6 Probability of winning the \$4 Powerball® lottery prize

Calculate the probability of winning the \$4 prize (matching one of the five white balls and the power ball).

Solution

We need to match one of the 5 winning white balls (therefore select 4 losing white balls) and match the one winning Powerball®.

$$\Pr(\text{one white and the power ball}) = \frac{\binom{5}{1}\cdot\binom{48}{4}\cdot\binom{1}{1}}{\binom{53}{5}\cdot\binom{42}{1}} = \frac{5\cdot 194{,}580\cdot 1}{2{,}869{,}685\cdot 42}$$

$$= \frac{972{,}900}{120{,}526{,}770} = \frac{1}{123.88} = .0081. \quad ❒$$

Example 1.4-7 Probability of winning nothing

Calculate the probability of losing one dollar (e.g., matching zero or one or two of the five white balls and not the power ball).

Solution

Losing options include matching 0, 1, or 2 winning white balls and not the power ball.

$$\Pr(\text{losing a dollar}) = \frac{\binom{5}{0}\cdot\binom{48}{5}\cdot\binom{41}{1}+\binom{5}{1}\cdot\binom{48}{4}\cdot\binom{41}{1}+\binom{5}{2}\cdot\binom{48}{3}\cdot\binom{41}{1}}{\binom{53}{5}\cdot\binom{42}{1}}$$

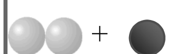
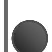
$$= \frac{1\cdot 1{,}712{,}304\cdot 41+5\cdot 194{,}580\cdot 41+10\cdot 17{,}296\cdot 41}{2{,}869{,}685\cdot 42} = \frac{117{,}184{,}724}{120{,}526{,}770}$$

$$= .9723$$

❐

Exercise 1-61 Check the remaining probabilities for the Powerball® game listed below. Verify that theses probabilities sum to 1.

Powerball® Probability Summary

Match	Prize	Odds
	Grand Prize	$\frac{1}{120{,}526{,}770}$
	$100,000	$\frac{41}{120{,}526{,}770}$
	$5,000	$\frac{240}{120{,}526{,}770}$
	$100	$\frac{9{,}840}{120{,}526{,}770}$
	$100	$\frac{11{,}280}{120{,}526{,}770}$
	$7	$\frac{462{,}480}{120{,}526{,}770}$
	$7	$\frac{172{,}960}{120{,}526{,}770}$
	$4	$\frac{972{,}900}{120{,}526{,}770}$
	$3	$\frac{1{,}712{,}304}{120{,}526{,}770}$
Nothing	$ 0	$\frac{117{,}184{,}724}{120{,}526{,}770}$

Exercise 1-62 Suppose that the Grand Prize is 50 million dollars and paid in cash today and that there are no taxes. Find the *average* winning computed as the sum of the possible winning times its probability.